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Integrable ladder t–J model with staggered shift of the spectral parameter

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Abstract

The generalization of the Yang–Baxter equations in the presence of $\mathbb{Z}_2$ grading along both chain and time directions is presented and an integrable model of t–J type with staggered disposition of shifts of the spectral parameter along the chain is constructed. The Hamiltonian of the model is computed in the fermionic formulation. It involves three neighbour site interactions and therefore can be considered as a zigzag ladder model. The algebraic Bethe ansatz technique is applied and the eigenstates as well as the eigenvalues of the transfer matrix of the model are found. It is argued that in the thermodynamic limit the lowest energy of the model is formed by the quarter filling of the states by fermions instead of the usual half filling.

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1. Introduction

Interest in ladder-type models first arose at the beginning of the 1990s (see [1] for a review) in connection with high-temperature superconductivity problems in metal oxides. It is believed that quasi-one-dimensional multi-ladder chains of strongly interacting electrons reflect the most important aspects of two-dimensional systems and can also reveal some properties of the weak coupling between conducting planes.

Recently there has been considerable interest in the construction of integrable ladder-type models motivated by the desire to use the powerful algebraic Bethe ansatz (ABA) technique [2, 3] in the exact investigations of the variety of physical phases of the models.

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In [5] integrable ladder models were constructed by extension of the symmetry algebra, and in [6] by first defining the ground state and then formulating a model which has it. The higher conservation laws of integrable models, which contain next-to-nearest-neighbour interactions, were used in the construction of ladder models in [7], developing the approach of [8].Models with alternating spins were considered in [9, 10]. There have also been some other attempts in this area [11, 12].

Usually, integrable models are homogeneous along the chain: namely, the spectral u and model parameters are the same in the product of R-matrices along the chain. It is obvious that, if one considers arbitrary shifts of the spectral parameters by some z_i in the monodromy matrix, we still have an integrable model. But in order to have a local Hamiltonian we need to consider shifts with fixed periodicity n, which causes the interaction of spins (or electrons) within an amount of n neighbours, leading to an n-ladder model. The staggered shift of the spectral parameter was first considered in [13] in an attempt to construct a relativistic invariant massive Tirring model in a specific limit of the homogeneous XXZ model. The inhomogeneous models were considered in [16–18].

In [4] we proposed an inhomogeneous model based on the XXZ spin chain, where the inhomogeneity appeared not only in the staggered shifts of the spectral parameter by some additional parameter θ, but also in change of structure of R-matrices in the product along the chain. Namely, two monodromy matrices of chains M_s, $s = 0, 1$ were considered along the time direction, where the R-matrices in the product have an alternating disposition of the anisotropy parameter $\pm\Delta$ of the XXZ model. In addition, and contrary to the case considered in [15–19], the spectral parameter of the second line has an opposite sign. Due to the double space translational invariance the Hamiltonian of the model contains interaction between three neighbours sites of the chain and therefore represents a zigzag-type ladder model. At the free fermionic point $\Delta = 0$ the model becomes a model of two noninteracting fermions, hopping separately in the odd and even sites of the chain.

In this paper we extend the construction of [4] to the ordinary t–J model [27–30], which, as was shown in [22], is the fermionized version of the spin-1 Uimin–Lai–Sutherland model [24–26]. Following [4], we consider two different expressions for monodromy matrices, which act on the states of the chain consecutively in time direction, and write two Yang–Baxter equations (YBEs) (see (2.14) and (2.15) below) for each step of alternating R-matrices along the chain. But as the solution shows, in this spin space isotropic model we have less modification of R-matrices of the chain than in the anisotropic XXZ model, being left only with the alternating shift of the spectral parameter and the change of sign of the spectral parameter in the second line. As we will see, though this gives us the same Bethe equations (BEs) for the spectral parameters of the excitations as for the model derived in [16–18], their energy and the energy of the ground state are different. It is argued that in the thermodynamic limit the lowest energy of the model is reached by the quarter filling of the states by fermions, instead of the usual half filling. The model contains an additional parameter θ.

In section 2 we formulate the model and find a local Hamiltonian, which has a zigzag ladder form. It consists of two chains with t–J-type Hamiltonians on each of them, the hopping term of electrons from one chain to the other and two types of interaction terms between chains. The first interaction term has the form of spin–spin interaction, where one spin is composed from two fermions on the same site of the chain, while the other spin is composed from two different fermions on the neighbour sites of the other chain of the ladder. The second interaction term has a topological form of interacting spins and is written for the triangles consisting of the zigzag rungs. As in the ordinary t–J model the present model also has global $gl(1|2)$ supersymmetry.

In section 3 we apply ABA in order to find the eigenvalues and the eigenstates of the model. At the end of section we find the ground state energy of the model and the spectrum of excitations in thermodynamic limit.

2. The Yang–Baxter equations and their solution

The key of integrability of the models is the YBE, which implies some restrictions on the $R_{ij}(u)$ matrix, the basic constituent of the monodromy matrix. The YBE ensures a local sufficient condition for the commutativity of the transfer matrices $\tau(u) = \operatorname{tr} T(u)$ at different values of the spectral parameter u, which corresponds to the rapidity of pseudo-particles of the model

$$[\tau(u), \tau(v)] = 0. \tag{2.1}$$

We use the fermionization technique (as an alternative to Jordan–Wigner transformation) developed in [21–23] (see also [20]) and work with the R-operators (rather than matrices) expressed in Fermi fields c_i, c_i^+, ($i = 1, \ldots, N$ is the chain site).

By definition R_{aj} acts as a intertwining operator on the space of direct product of the so-called auxiliary $V_a(v)$ and quantum $V_j(u)$ spaces

$$R_{aj}(u, v): \quad V_a(u) \otimes V_j(v) \to V_j(v) \otimes V_a(u). \tag{2.2}$$

The spaces $V_a(u)$ and $V_j(v)$ with spectral parameters u and v are irreducible representations of the affine quantum algebra $\mathcal{U}_q(\hat{g})$, which is the symmetry algebra of the integrable model under consideration. Provided that the states $|a\rangle \in V_a$ and $|j\rangle \in V_j$ form a basis for the spaces V_a and V_j, following [21] we can represent the action of the operator R_{aj} as

$$R_{aj}|j_1\rangle \otimes |a_1\rangle = \left(R_{aj}\right)_{a_1 j_1}^{a_2 j_2} |a_2\rangle \otimes |j_2\rangle \tag{2.3}$$

where the summation is over the repeated indices a_2 and j_2 (but not over a_1 and j_1).

By introducing the Hubbard operators

$$X_{a_2}^{a_1} = |a_2\rangle\langle a_1| \qquad X_{j_2}^{j_1} = |j_2\rangle\langle j_1| \tag{2.4}$$

in the graded spaces V_a and V_j correspondingly, one can rewrite (2.3) as

$$\begin{aligned} R_{aj} = R_{aj}|j_1\rangle|a_1\rangle\langle a_1|\langle j_1| &= \left(R_{aj}\right)_{a_1 j_1}^{a_2 j_2} |a_2\rangle|j_2\rangle\langle a_1|\langle j_1| \\ &= (-1)^{p(a_1)p(j_2)} \left(R_{aj}\right)_{a_1 j_1}^{a_2 j_2} X_{a_2}^{a_1} X_{j_2}^{j_1} \end{aligned} \tag{2.5}$$

where the sign factor takes into account the possible gradings of the states $|a_i\rangle$ and $|j_i\rangle$, and $p(a_i)$ and $p(j_i)$ denote the corresponding parities and summation over the repeated indices.

In terms of operators R_{ij} the matrix-valued YBE can be written in the following operator form:

$$R_{ab}(u, v) R_{aj}(u, w) R_{bj}(v, w) = R_{bj}(v, w) R_{aj}(u, w) R_{ab}(u, v). \tag{2.6}$$

Let us now consider $\mathbb{Z}_2$ graded quantum $V_{j,\rho}(v)$ and auxiliary $V_{a,\sigma}(u)$ spaces, $\rho, \sigma = 0, 1$. In this case we have 4×4 R-matrices, which act on the direct product of the spaces $V_{a,\sigma}(u)$ and $V_{j,\rho}(v)$, ($\sigma, \rho = 0, 1$), mapping them on the intertwined direct product of $V_{a,\bar{\sigma}}(u)$ and $V_{j,\bar{\rho}(v)}$ with the complementary $\bar{\sigma} = (1 - \sigma)$, $\bar{\rho} = (1 - \rho)$ indices

$$R_{aj,\sigma\rho}(u, v): \quad V_{a,\sigma}(u) \otimes V_{j,\rho}(v) \to V_{j,\bar{\rho}}(v) \otimes V_{a,\bar{\sigma}}(u). \tag{2.7}$$

It is convenient to introduce two transmutation operations ι_1 and ι_2 with the property $\iota_1^2 = \iota_2^2 = \mathrm{id}$ for the quantum and auxiliary spaces correspondingly, and to mark the operators $R_{aj,\sigma\rho}$ as follows:

$$\begin{aligned} R_{aj,00} &\equiv R_{aj} \qquad & R_{aj,01} &\equiv R_{aj}^{\iota_1} \\ R_{aj,10} &\equiv R_{aj}^{\iota_2} \qquad & R_{aj,11} &\equiv R_{aj}^{\iota_1\iota_2}. \end{aligned} \tag{2.8}$$

The introduction of the $\mathbb{Z}_2$ grading in quantum space means that we now have two monodromy matrices M_ρ, $\rho = 0, 1$, which act on the space $V_\rho = \prod_{j=1}^{N} V_{j,\rho}$ by mapping it on $V_{\bar\rho} = \prod_{j=1}^{N} V_{j,\bar\rho}$

$$M_\rho : V_\rho \to V_{\bar\rho} \qquad \rho = 0, 1. \tag{2.9}$$

It is now clear that the monodromy matrix of the model, which should define the partition function, is the product of two monodromy matrices

$$M(u) = M_0(u) M_1(u). \tag{2.10}$$

Now, because of the grading in the auxiliary space, we would like to construct the monodromy matrices $M_{0,1}$ as a staggered product of the R_{aj} and $\bar R^{\iota_2}_{aj}$ matrices. Let us define

$$\begin{aligned} M_0(u) &= \prod_{j=1}^{N} R_{a,2j-1}(u) \bar R^{\iota_2}_{a,2j}(u) \\ M_1(u) &= \prod_{j=1}^{N} \bar R^{\iota_1}_{a,2j-1}(u) R^{\iota_1\iota_2}_{a,2j}(u) \end{aligned} \tag{2.11}$$

where the notation $\bar R$ in general means the different parametrization of the R-matrix via model (λ) and spectral (u) parameters and can be considered as an operation over R with property $\bar{\bar R} = R$.

In order to have an integrable model with commuting transfer matrices (2.1) for different spectral parameters

$$[\operatorname{tr} M(u), \operatorname{tr} M(v)] = 0 \tag{2.12}$$

it is enough to have the following relations for the $\tau_\sigma(u) = \operatorname{tr} M_\sigma(u)$, ($\sigma = 0, 1$):

$$\tau_\sigma(\lambda, u)\ \tau_{1-\sigma}(\lambda, v) = \bar\tau_\sigma(\lambda, v)\ \bar\tau_{1-\sigma}(\lambda, u) \qquad \sigma = 0, 1. \tag{2.13}$$

It is not hard to see that, in order to ensure the commutativity condition (2.12), the R- and $\bar R$-matrices in (2.11) should fulfil the following two YBEs, which in the so-called check formalism defined by operator $\check R_{ij} = R_{ij} P_{ij}$ (P_{ij} is the permutation operator) have the form

$$\check R_{12}(u, v) \check{\bar R}^{\iota_1}_{23}(u) \check R_{12}(v) = \check R^{\iota_1}_{23}(v) \check{\bar R}_{12}(u) \check{\bar R}_{23}(u, v) \tag{2.14}$$

$$\check{\bar R}_{12}(u, v) \check R^{\iota_1\iota_2}_{23}(u) \check{\bar R}^{\iota_2}_{12}(v) = \check{\bar R}^{\iota_1\iota_2}_{23}(v) \check R^{\iota_2}_{12}(u) \check R_{23}(u, v). \tag{2.15}$$

It is also convenient to use the fermionic (graded) operator formalism for R-matrices and the YBEs developed in [20–23].

We are going to extend the construction described above to the t–J model [27–29]. As was shown in [22], the R-operator of the t–J model can be obtained from the spin-1 model of [24–26] by fermionization (an alternative approach to the Jordan–Wigner transformation) of their spin-1 R-matrix. A minimum of two sorts of fermions is needed in order to express three basic states $|+\rangle$, $|0\rangle$, $|-\rangle$ of the spin-1 particle with the z component of the spin equal to $1, 0, -1$ correspondingly.

Now let us define ${c_\sigma}^+$, c_σ, where $\sigma = \uparrow\downarrow$, as creation–annihilation operators of fermions with the up and down spins respectively, together with their Fock space $|0\rangle$,$|\sigma\rangle$.

The states with definite third projection of the algebra $SU(2)$ can be realized through fermionic Fock space as follows:

$$|-\rangle \equiv |0, \downarrow\rangle, |+\rangle \equiv |\uparrow, 0\rangle, |0\rangle \equiv |0, 0\rangle \tag{2.16}$$

numerated as $|1\rangle$, $|2\rangle$, $|3\rangle$ respectively.

As is obvious from (2.16), we have constructed a graded space with the following parities for the basic vectors:

$$p(|+\rangle) = p(|-\rangle) = 1 \qquad p(|0\rangle) = 0. \tag{2.17}$$

In order to proceed further and write the fermionic R-matrix we should calculate the Hubbard operator $X_m^n = |m\rangle\langle n|$; $m, n = 1, 2, 3$

$$X_m^k = \begin{pmatrix} |-\rangle\langle -| & |-\rangle\langle +| & |-\rangle\langle 0| \\ |+\rangle\langle -| & |+\rangle\langle +| & |+\rangle\langle 0| \\ |0\rangle\langle -| & |0\rangle\langle +| & |0\rangle\langle 0| \end{pmatrix} = \begin{pmatrix} (1-n_\uparrow)n_\downarrow & c_\downarrow^+ c_\uparrow & (1-n_\uparrow)c_\downarrow^+ \\ c_\uparrow^+ c_\downarrow & n_\uparrow(1-n_\downarrow) & c_\uparrow^+(1-n_\downarrow) \\ (1-n_\uparrow)c_\downarrow & c_\uparrow(1-n_\downarrow) & (1-n_\uparrow)(1-n_\downarrow) \end{pmatrix}. \tag{2.18}$$

The trace of this operator is

$$\Delta = X_m^m = 1 - n_\uparrow n_\downarrow \tag{2.19}$$

which is an identity operator on the space of states, where the double occupancy of the sites by fermions is excluded.

Following [22] let us write down the fermionic R-operator for the t–J (spin 1 [24–26]) model:

$$\check{R}_{i,j}(u) = a(u)I_{i,j} + b(u)\Pi_{i,j} = a(u)I_{i,j} + b(u)\sum_{m,n=1}^{N}(-1)^{p(m)}X_i{}_n^m X_j{}_m^n \tag{2.20}$$

where $\Pi_{i,j}$ is the graded permutation operator of the spaces V_i and V_J.

Now by putting the R-matrix form (2.20) into the YBEs (2.14) and (2.15) and after some calculations one can find 12 equations which require the operations $\tilde{}$ and transformation ι_1 to be

$$\begin{aligned} &\tilde{a}(u,v) = a(u,v) \qquad \tilde{b}(u,v) = b(u,v) \\ &\frac{a^{\iota_1}(u)}{b^{\iota_1}(u)} = \frac{a(u)}{b(u)}. \end{aligned} \tag{2.21}$$

Conditions (2.21) reduce the 12 equations to the following two equations:

$$\begin{aligned} &a(u,v)[\bar{a}(u)b(v) - \bar{b}(u)a(v)] + b(u,v)\bar{a}(u)a(v) = 0 \\ &a(u,v)[a^{\iota_2}(u)\bar{b}^{\iota_2}(v) - b^{\iota_2}(u)\bar{a}^{\iota_2}(v)] + b(u,v)a^{\iota_2}(u)\bar{a}^{\iota_2}(v) = 0 \end{aligned} \tag{2.22}$$

the consistency condition of which can be found easily as follows:

$$\frac{b(v)}{a(v)} - \frac{\bar{b}^{\iota_2}(v)}{\bar{a}^{\iota_2}(v)} = \frac{\bar{b}(u)}{\bar{a}(u)} - \frac{b^{\iota_2}(u)}{a^{\iota_2}(u)} = \text{constant} = \theta. \tag{2.23}$$

Here θ is the new parameter of our model.

Then the solution of (2.22) for the intertwiner parameters $a(u, v)$ and $b(u, v)$ is

$$\frac{b(u,v)}{a(u,v)} = \frac{\bar{b}(u)}{\bar{a}(u)} - \frac{b(v)}{a(v)}. \tag{2.24}$$

We should now define the ι_2 operation. It is easy to see from (2.23) for $v = u$ that the ι_2 operation can be consistently defined as follows:

$$\frac{b^{\iota_2}(u)}{a^{\iota_2}(u)} = -\frac{b(u)}{a(u)} \qquad \frac{\bar{b}^{\iota_2}u}{\bar{a}^{\iota_2}(u)} = -\frac{\bar{b}(u)}{\bar{a}(u)}. \tag{2.25}$$

It is clear from (2.22)–(2.24) that the ratio $\frac{b(u)}{a(u)}$ can be taken as a spectral parameter $\frac{b(u)}{a(u)} = u$, $\frac{\bar{b}(u)}{\bar{a}(u)} = \bar{u} = \theta - u$.

Finally, after appropriate normalization of $a(u)$ and $b(u)$ in order to have $a(u)+b(u)=1$, one finds the following solution of YBEs:

$$\begin{array}{lll} a(u)=\dfrac{1}{1+u} & \bar{a}(u)=a(\bar{u})\dfrac{1}{1+\theta-u} & a(u,v)=a(v-u)=\dfrac{1}{1+v-u} \\ b(u)=\dfrac{u}{1+u} & \bar{b}(u)=b(\bar{u})=\dfrac{\theta-u}{1+\theta-u} & b(u,v)=b(v-u)=\dfrac{v-u}{1+v-u}. \end{array} \tag{2.26}$$

According to the standard prescription of the ABA technique the logarithmic derivative of the transfer matrix at some point defines the Hamiltonian of the model

$$H=-\left.\frac{\partial \ln \tau(u)}{\partial u}\right|_{u=0}. \tag{2.27}$$

As is known, in order for the Hamiltonian to be local, it is necessary to have a value u_0 such that

$$\check{R}_{i,j}(u_0)=I_{i,j}. \tag{2.28}$$

Analysing the solutions of our YBEs (2.26), one can see from (2.20) that, at the point $u_0=0$, only $\check{R}_{i,j}(0)=I_{i,j}$ and $\check{\bar{R}}_{i,j}(0)\neq I_{i,j}$. As calculations show, the Hamiltonian is nevertheless local, but it contains interaction between four neighbour points.

Technically, in order to calculate the logarithmic derivative (2.27), one should put the expression of $\check{R}_{i,j}$ operators around $u_0=0$ up to linear terms

$$\begin{aligned} &\check{R}_{i,j}=I_{i,j}+uH_{i,j} \\ &\check{\bar{R}}_{i,j}(u)=R_{i,j}(\theta)-uH_{i,j} \\ &\check{R}^{t_2}_{i,j}(u)=I_{i,j}-uH_{i,j} \\ &\check{\bar{R}}^{t_2}_{i,j}(u)=R_{i,j}(-\theta)+uH_{i,j} \end{aligned} \tag{2.29}$$

with

$$H_{i,j}=\sum_{m,n}(-1)^{p(m)}X_i^{\ m}_{\ n}X_j^{\ n}_{\ m} \tag{2.30}$$

into the expression (2.10) of the monodromy matrix

$$M(u)=M_0(u)M_1(u)=\ldots\bar{R}^{t_2}_{01}(u)R_{12}(u)\bar{R}^{t_2}_{23}R_{34}(u)\ldots R^{t_1t_2}_{12}(u)\bar{R}^{t_1}_{23}(u)R^{t_1t_2}_{34}(u)\bar{R}^{t_1}_{45}\ldots. \tag{2.31}$$

As a result, after some algebraic calculations, we obtain the following Hamiltonian for the present staggered t–J model:

$$\begin{aligned} H=\theta\Delta\sum_{i=1}^{N}\Bigg\{ &\sum_{\sigma=\uparrow\downarrow}\left(2-\frac{n_{i-1}}{2}-\frac{n_{i-2}}{2}\right)(c^+_{i,\sigma}c_{i+1,\sigma}-c^+_{i+1,\sigma}c_{i,\sigma}) \\ &+\sum_{\sigma=\uparrow\downarrow}\left[\left(1-\frac{n_{i+1}}{2}+(-1)^i\theta\right)c^+_{i+2,\sigma}c_{i,\sigma}-\left(1-\frac{n_{i+1}}{2}-(-1)^i\theta\right)c^+_{i,\sigma}c_{i+2,\sigma}\right] \\ &+2\Bigg[(\vec{S}_{i+2}+\vec{S}_{i-1})(\vec{S}_{i,i+1}-\vec{S}_{i+1,i})+\vec{S}_{i+1}(\vec{S}_{i+2,i}-\vec{S}_{i,i+2}) \\ &+\theta(-1)^i\left(\vec{S}_{i-1}\vec{S}_{i+1}-\frac{1}{4}n_{i-1}n_{i+1}+\frac{n_{i-1}+n_{i+1}}{2}\right) \\ &-\mathrm{i}\epsilon^{abc}S^a_iS^b_{i+1}S^c_{i+2}\Bigg]\Bigg\}\Delta \end{aligned} \tag{2.32}$$

where $\Delta=\prod_{i=1}^{N}\Delta_i=\prod_{i=1}^{N}(1-n_{i\uparrow}n_{i\downarrow})$ is the projector which excludes the double occupancy by electrons at any site i.

The spin operators $\vec{S}_i$ and $\vec{S}_{i,j}$ are defined as follows:

$$\begin{aligned}\vec{S}_i &= \tfrac{1}{2}\Psi^+_{i,\beta}\vec{\sigma}^\beta_\alpha\Psi^\alpha_i = \tfrac{1}{2}\Delta_i c^+_{i,\alpha}\vec{\sigma}^\alpha_\beta c^\beta_i\Delta_i \\ \vec{S}_{i,j} &= \tfrac{1}{2}\Psi^+_{i,\beta}\vec{\sigma}^\beta_\alpha\Psi^\alpha_j = \tfrac{1}{2}\Delta_i c^+_{i,\alpha}\vec{\sigma}^\alpha_\beta c^\beta_j\Delta_j\end{aligned} \tag{2.33}$$

where

$$\Psi^1 = (1-n_\uparrow)c_\downarrow \qquad \Psi^2 = c_\uparrow(1-n_\downarrow) \tag{2.34}$$

and $\vec{\sigma}$ are Pauli matrices.

3. Algebraic Bethe ansatz for the staggered t–J model

In this section we apply the technique of ABA [3, 28, 29] to the present model and find the eigenvalues and eigenstates of the Hamiltonian (2.32).

For this purpose let us introduce the L operators as follows:

$$(L_{i,j})^{k'}_k = \langle k|R_{i,j}|k'\rangle \tag{3.1}$$

which is a matrix in the horizontal auxiliary space and an operator in the quantum space. In matrix form it looks like

$$L_{i,j} = \begin{pmatrix} a(u)-b(u)(1-n_\uparrow)n_\downarrow & -b(u)c_\uparrow c^+_\downarrow & b(u)(1-n_\uparrow)c^+_\downarrow \\ -b(u)c^+_\uparrow c_\downarrow & a(u)-b(u)n_\uparrow(1-n_\downarrow) & b(u)c^+_\uparrow(1-n_\downarrow) \\ b(u)(1-n_\uparrow)c_\downarrow & b(u)c_\uparrow(1-n_\downarrow) & a(u)+b(u)(1-n_\uparrow)(1-n_\downarrow) \end{pmatrix}. \tag{3.2}$$

The monodromy matrix $M^k_{k'}(u)$, which is defined by the matrix elements of the monodromy operators (2.10), (2.11) in the auxiliary space can be expressed as a product of $L_{i,j}$ matrices as follows:

$$\begin{aligned}M_0(u)^k_{k'} &= \langle k|M_0(u)|k'\rangle = (-1)^{p(k)p(k')}(\bar{L}^{\iota_2}_{01})^k_{k_1}(L_{02})^{k_1}_{k_2}\ldots(L_{0N})^{k_{N-1}}_{k'} \\ M_1(u)^k_{k'} &= \langle k|M_1(u)|k'\rangle = (-1)^{p(k)p(k')}(L^{\iota_1\iota_2}_{01})^k_{k_1}(\bar{L}^{\iota_1}_{02})^{k_1}_{k_2}\ldots(\bar{L}^{\iota_1}_{0N})^{k_{N-1}}_{k'}.\end{aligned} \tag{3.3}$$

Following the notation of [29], one can write

$$M_s(u)^k_{k'} = \begin{pmatrix} A_{s,11}(u) & A_{s,12}(u) & B_{s,1}(u) \\ A_{s,12}(u) & A_{s,22}(u) & B_{s,2}(u) \\ C_{s,1}(u) & C_{s,2}(u) & D_s(u) \end{pmatrix} \qquad s=0,1 \tag{3.4}$$

where $A_{s,ab}$, $B_{s,a}$, $C_{s,a}$, D_s; $(a,b=1,2)$ act on the quantum space.

Then, as a super-trace of the monodromy matrix (3.4) the transfer matrix $\tau(u)$ has the form

$$\tau_s(u) = -A_{s,11}(u) - A_{s,22}(u) + D_s(u) \qquad s=0,1. \tag{3.5}$$

The matrix elements of the monodromy matrix obey the algebraic relations

$$\begin{aligned}&(-1)^{p(k'')(p(m')+p(m''))}\check{R}^{km}_{k'm'}(u,v)M^{m'}_{1,m''}(u)M^{k'}_{0,k''}(v) \\ &\quad= (-1)^{p(k')(p(m)+p(m'))}M^m_{1,m'}(v)M^k_{0,k'}(u)\check{R}^{k'm'}_{k''m''}(u,v) \\ &(-1)^{p(k'')(p(m')+p(m''))}\check{R}^{km}_{k'm'}(u,v)M^{m'}_{0,m''}(u)M^{k'}_{1,k''}(v) \\ &\quad= (-1)^{p(k')(p(m)+p(m'))}M^m_{0,m'}(v)M^k_{1,k'}(u)\check{R}^{k'm'}_{k''m''}(u,v)\end{aligned} \tag{3.6}$$

which are the consequence of YBEs (2.14), (2.15). In getting (3.6) we have used the properties (2.21).

Consider now the empty fermionic state as a test 'vacuum'

$$|\Omega\rangle_s = |0,0,\ldots,0\rangle_s = |0\rangle_{1s}|0\rangle_{2s}\ldots|0\rangle_{Ns} \qquad s=0,1 \tag{3.7}$$

and let us check that $|\Omega\rangle$ is indeed an eigenstate of transfer matrix (3.5):

$$\tau_s(u)|\Omega\rangle_s = \nu_s^{(0)}|\Omega\rangle_{1-s}. \tag{3.8}$$

From (3.3), $\tau_s(u)$ is a product of $L_{i,j}$ matrices. Hence, in order to check (3.8) we should first calculate $L_{0k}|0\rangle_{s,k}$. It appears that

$$\bar{L}_{0k}^{\iota_1}|0\rangle_{s,k} = \begin{pmatrix} \bar{b}(u)^{\iota_1} & 0 & \bar{b}^{\iota_1}(u)c_{k\downarrow}^{+} \\ 0 & \bar{b}^{\iota_1}(u) & \bar{b}^{\iota_1}(u)c_{k\uparrow}^{+} \\ 0 & 0 & \bar{a}^{\iota_1}(u) + \bar{b}^{\iota_1}(u) \end{pmatrix} |0\rangle_k \qquad s = 0, 1. \tag{3.9}$$

We see that $L_{0k}|0\rangle_k$ is an upper-triangular matrix. Therefore the action of the product of L_{0k} in (3.3) on vacuum $|\Omega\rangle_k$ as a matrix also has an upper-triangular form

$$\begin{aligned} M_1(u)_{k'}^{k}|\Omega\rangle_1 &= \begin{pmatrix} [b^{\iota_2\iota_1}(u)\bar{b}^{\iota_1}(u)]^{\frac{N}{2}} & 0 & B_{1,1}(u) \\ 0 & [b^{\iota_2\iota_1}(u)\bar{b}^{\iota_1}(u)]^{\frac{N}{2}} & B_{1,2}(u) \\ 0 & 0 & 1 \end{pmatrix} |\Omega\rangle_1 \\ M_0(u)_{k'}^{k}|\Omega\rangle_0 &= \begin{pmatrix} [\bar{b}^{\iota_2}(u)b(u)]^{\frac{N}{2}} & 0 & B_{0,1}(u) \\ 0 & [\bar{b}^{\iota_2}(u)b(u)]^{\frac{N}{2}} & B_{0,2}(u) \\ 0 & 0 & 1 \end{pmatrix} |\Omega\rangle_0 \end{aligned} \tag{3.10}$$

where we have used that $a(u) + b(u) = 1$.

We see that the $B_{s,1}(u)$ and $B_{s,2}(u)$, $(s = 0, 1)$ operators create one-particle states while $C_{s,1}(u)$, $C_{s,2}(u)$ operators annihilate them:

$$C_{s,a}(u)|\Omega\rangle_s = 0 \qquad s = 0, 1 \quad a = 1, 2. \tag{3.11}$$

We see from (3.10) that

$$\nu_s^{(0)}(u) = 1 - 2[b^{\iota_2}(u)\bar{b}(u)]^{\frac{N}{2}} \tag{3.12}$$

where $b^{\iota_2}(u)$ and $\bar{b}(u)$ are defined by (2.26) and N is the length of the chain.

This observation leads us to the following ansatz for the eigenstates of $\tau(v)$:

$$|v_1, v_2, \ldots, v_n|F\rangle_0 = F^{a_n \ldots a_1} B_{0,a_1}(v_1)B_{1,a_2}(v_2)\ldots B_{0,a_n}(v_n)|\Omega\rangle_0 \qquad a_i = 1, 2 \tag{3.13}$$

is an n particle state. The $F^{a_n \ldots a_1}$ is a function of spectral parameters v_j to be specified later.

The action of the transfer matrix (3.5) on the states (3.13) is determined by the relations (3.10) and the intertwining properties of the $A_{s,ab}(u)$, $D_s(u)$, $B_{s,a}(u)$ operators are defined from the YBE (3.6). The components of the intertwining relations, which we need for the construction of the ABA are

$$\begin{aligned} D_1(u)B_{0,a}(v) &= \frac{1}{b(u,v)}B_{1,a}(v)D_0(u) - \frac{a(v,u)}{b(v,u)}B_{1,a}(u)D_0(v) \\ A_{1,ba}(u)B_{0,c}(v) &= \frac{r_{bc}^{b'c'}(u,v)}{b(u,v)}B_{1,c'}(v)A_{0,b'a}(u) + \frac{a(u,v)}{b(u,v)}B_{1,b}(u)A_{0,ca}(v) \\ B_{1,a}(u)B_{0,b}(v) &= r_{ab}^{b'a'}(u,v)B_{1,a'}(v)B_{0,b'}(u) \end{aligned} \tag{3.14}$$

where

$$r_{bc}^{b'c'}(v) = -a(v)\delta_b^{c'}\delta_c^{b'} + b(v)\delta_b^{b'}\delta_c^{c'} = -a(v)I_{bc}^{c'b'} - b(v)\Pi_{bc}^{(1)b'c'}. \tag{3.15}$$

Here $\Pi_{bc}^{(1),b'c'}$ is a graded permutation operator for $p(1) = p(2) = 1$, one can check that it fulfils the following YBE:

$$r(\lambda-\mu)_{a_3c_3}^{a_2c_2}r(\lambda)_{c_2d_2}^{a_1b_1}r(\mu)_{a_2c_2}^{d_2b_2} = r(\mu)_{a_2c_2}^{a_1c_1}r(\lambda)_{a_3b_3}^{c_2d_2}r(\lambda-\mu)_{d_2b_2}^{c_1b_1}. \tag{3.16}$$

Now by use of (3.14), we can obtain that the diagonal elements of the monodromy matrix act on the states (3.13) as follows:

$$\begin{aligned} D_1(u)\Big|v_1,\dots,v_n|F\rangle_1 &= \prod_{j=1}^{n}\frac{1}{b(v_j,u)}\Big|v_1,\dots,v_n|F\rangle_0 \\ &\quad+\sum_{k=1}^{n}(\tilde{\Lambda}_k)^{b_1\dots b_n}_{a_1\dots a_n}B_{1,b_k}(u)F^{a_n\dots a_1}\prod_{j=1,\,j\neq k}^{n}B_{b_j}(v_j)|\Omega\rangle_0 \\ \big[A_{1,11}(u)+A_{1,22}(u)\big]\mid v_1,\dots,v_n|F\rangle_1 & \\ =\ -\prod_{i=1}^{n}\frac{1}{b(u,v_i)}&[b^{\iota_1\iota_2}(u)\bar{b}^{\iota_2}(u)]^{\frac{N}{2}}\tau^{(1)a'_1\dots a'_n}_{a_1\dots a_n}(u)F^{a_n\dots a_1}\prod_{i=1}^{n}B_{a'_i}(v_i)|\Omega\rangle_0 \\ &\quad+\sum_{k=1}^{n}(\Lambda_k)^{b_1\dots b_n}_{a_1\dots a_n}F^{a_n\dots a_1}B_{1,b_k}(u)\prod_{i=1,\,j\neq k}^{n}B_{b_j}(v_j)|\Omega\rangle_0 \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} \tau^{(1)a'_1\dots a'_n}_{a_1\dots a_n}(u) &= -r^{b_1a'_1}_{ca_1}(u,v_1)\dots r^{ca'_n}_{b_{n-1}a_n}(u,v_n) \\ &= \mathrm{str}[l_n(u,v_n)l_{n-1}(u,v_{n-1})\dots l_1(u,v_1)] \end{aligned} \tag{3.18}$$

and

$$[l_k(u,v_k)]^{b_k}_{b_{k-1}} = r^{b_ka'_k}_{b_{k-1}a_k}(u,v_k). \tag{3.19}$$

As follows from (3.19), l_k is a 2×2 matrix, the elements of which are the operators

$$l_k(u)=\begin{pmatrix} l^1_{k,1} & l^2_{k,1}\\ l^1_{k,2} & l^2_{k,2}\end{pmatrix}=\begin{pmatrix} b(u)I-a(u)e^1_1 & -a(u)e^2_1\\ -a(u)e^1_2 & b(u)I-a(u)e^2_2\end{pmatrix} \tag{3.20}$$

where e^b_a are quantum operators in the nth space with matrix representation $(e^b_a)^\alpha_\beta=\delta^\alpha_a\delta^b_\beta$.

It is obvious that the eigenvalue condition

$$(D_{s,1}(u)-A_{s,11}(u)-A_{s,22}(u))|v_1,\dots,v_n|F\rangle_s = \nu_s(u,v_1,\dots,v_n)|v_1,\dots,v_n|F\rangle_{1-s} \tag{3.21}$$

is fulfilled if:

(i) we impose the cancellation of unwanted terms in (3.17)

$$[(\tilde{\Lambda}_k)^{b_1\dots b_n}_{a_1\dots a_n}-(\Lambda_k)^{b_1\dots b_n}_{a_1\dots a_n}]F^{a_n\dots a_1}=0 \tag{3.22}$$

called BE, and

(ii) we solve the eigenvalue problem for the small transfer matrix (3.18)

$$\tau^{(1)a'_1\dots a'_n}_{a_1\dots a_n}(u;v_1,\dots,v_n)F^{a_1\dots a_n}=\nu^{(1)}(u;v_1,\dots,v_n)F^{a'_1\dots a'_n} \tag{3.23}$$

and then we have the following expression for eigenvalues:

$$\begin{aligned} \nu_1(u;v_1,\dots,v_n)&=\prod_{i=1}^{n}\frac{1}{b(v_i,u)}+[b^{\iota_1\iota_2}(u)\bar{b}^{\iota_1}(u)]^{\frac{N}{2}}\prod_{j=1}^{n}\frac{1}{b(u,v_j)}\nu^{(1)}(u,v_i)\\ \nu_0(u;v_1,\dots,v_n)&=\prod_{i=1}^{n}\frac{1}{b(v_i,\bar{u})}+[b^{\iota_1}(\bar{u})\bar{b}^{\iota_1}(\bar{u})]^{\frac{N}{2}}\prod_{j=1}^{n}\frac{1}{b(\bar{u},v_j)}\nu^{(1)}(\bar{u},v_i). \end{aligned} \tag{3.24}$$

For the solution of the second equation (ii) we should make the ABA for a small auxiliary problem of a chain, with length n (number of particles) and 'nested' transfer matrix $\tau^{(1)a'_1\dots a'_n}_{a_1\dots a_n}(u;v_1,\dots,v_n)$. This is why this procedure is called nested algebraic Bethe ansatz (NABA). In [29] it was demonstrated how to calculate $\Lambda^{b_1\dots b_n}_{a_1\dots a_n}$ and $\tilde{\Lambda}^{b_1\dots b_n}_{a_1\dots a_n}$ and to reduce the condition of cancellation of the unwanted terms for the ordinary t–J model to some

equation. It is not necessary to repeat the same calculation here since it differs very little from the one carried out, the only difference being in the term

$$(A_{1,11} + A_{1,22})|\Omega\rangle = [b^{\iota_1\iota_2}(u)\bar{b}^{\iota_1}(u)]^{\frac{N}{2}}|\Omega\rangle. \tag{3.25}$$

Therefore we obtain the following conditions:

$$\tau^{(1)b'_1\ldots b'_n}_{b_1\ldots b_n}(v_k|v_1,\ldots,v_n)F^{b_n\ldots b_1} = [b^{\iota_1\iota_2}(v_k)\bar{b}^{\iota_1}(v_k)]^{-\frac{N}{2}}\prod_{i=1,i\neq k}^{n}\frac{b(v_k v_i)}{b(v_i v_k)}F^{b'_n\ldots b'_1} \tag{3.26}$$

as the BE.

In the next step of the NABA we want to find the eigenvalues and eigenstates of $\tau^{(1)}(u)$. It is clear from (3.16) that we have another small integrable model with the R-matrix $r^{a'b'}_{ab}(u)$ defined by (3.15) and the corresponding transfer matrix $\tau^{(1)}(u)$.

Therefore, we should apply a non-ordinary ABA to this problem. The YBE for the problem is

$$r^{a'b'}_{ab}(u,v)\hat{M}^{(1)a''}_{a'}(u)\hat{M}^{(1)b''}_{b'}(v) = \hat{M}^{(1)b'}_{b}(v)\hat{M}^{(1)a'}_{a}(u)r^{a''b''}_{a'b'}(u,v) \tag{3.27}$$

where $M^{(1)a'}_{a}$ is the corresponding (nested) monodromy matrix.

Now if we define

$$M^{(1)}(u) = \begin{pmatrix} A^{(1)}(u) & B^{(1)}(u) \\ C^{(1)}(u) & D^{(1)}(u) \end{pmatrix} \qquad \tau^{(1)}(u) = -A^{(1)}(u) - D^{(1)}(u) \tag{3.28}$$

then by use of (3.15) and YBE (3.27) we find

$$\begin{aligned} D^{(1)}(u)B^{(1)}(v) &= \frac{1}{b(u,v)}B^{(1)}(v)D^{(1)}(u) + \frac{a(v,u)}{b(v,u)}B^{(1)}(u)D^{(1)}(v) \\ A^{(1)}(u)B^{(1)}(v) &= \frac{a(u,v)}{b(u,v)}B^{(1)}(u)A^{(1)}(v) + \frac{1}{b(v,u)}B^{(1)}(v)A^{(1)}(u) \\ B^{(1)}(u)B^{(1)}(v) &= B^{(1)}(v)B^{(1)}(u). \end{aligned} \tag{3.29}$$

Let us take as reference state

$$\begin{aligned} |0\rangle^{(1)}_k &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |\Omega\rangle^{(1)} &= |0\rangle^{(1)}_1\ldots|0\rangle^{(1)}_n = \bigotimes_{k=1}^{n}|0\rangle^{(1)}_k. \end{aligned} \tag{3.30}$$

The action of the nested monodromy matrix $M^{(1)}(u)$ on the reference state $|\Omega\rangle^{(1)}$ is described by the action $l_k(u)$ on $|0\rangle^{(1)}_k$, which we can find from (3.20). So we obtain

$$\begin{aligned} A^{(1)}(u)|\Omega\rangle^{(1)} &= \prod_{i=1}^{n}[b(u,v_i) - a(u,v_i)]|\Omega\rangle^{(1)} = \prod_{i=1}^{n}\frac{b(u,v_i)}{b(v_i,u)}|\Omega\rangle^{(1)} \\ D^{(1)}(u)|\Omega\rangle^{(1)} &= \prod_{i=1}^{n}b(u,v_j)|\Omega\rangle^{(1)}. \end{aligned} \tag{3.31}$$

For the eigenstates of $\tau^{(1)}(v)$, we have the following ansatz:

$$|v^{(1)}_1,\ldots,v^{(1)}_m\rangle = B^{(1)}(v^{(1)}_1)B^{(1)}(v^{(1)}_2)\ldots B^{(1)}(v^{(1)}_m)|\Omega\rangle^{(1)}. \tag{3.32}$$

The action of $\tau^{(1)}(u)$ on the states (3.32) is the same as the action of the diagonal elements of (3.28) on those states. By use of (3.14) we obtain

$$D^{(1)}(u)|v_1^{(1)},\dots,v_m^{(1)}\rangle = \prod_{j=1}^{n} b(u,v_j)|v_1^{(1)},\dots,v_m^{(1)}\rangle + \sum_{k=1}^{m}\Lambda_k^{(1)}B^{(1)}(u)\prod_{i=1,i\neq k}^{m} B^{(1)}(v_i)|\Omega\rangle^{(1)}$$

$$A^{(1)}(u)|v_1^{(1)},\dots,v_m^{(1)}\rangle = \prod_{i=1}^{m}\frac{1}{b(v_i^{(1)},u)}\prod_{j=1}^{n}\frac{b(u,v_j)}{b(v_j,u)}|v_1^{(1)},\dots,v_m^{(1)}\rangle \tag{3.33}$$

$$+\sum_{k=1}^{m}\tilde{\Lambda}_k^{(1)}B^{(1)}(u)\prod_{j=1,j\neq k}^{} B^{(1)}(v_j)|\Omega\rangle^{(1)}.$$

From (3.33) we can easily write the eigenvalues of $\tau^{(1)}(u)$:

$$\tau^{(1)}|v_1^{(1)},\dots,v_m^{(1)}\rangle = -\Bigg[\prod_{i=1}^{m}\frac{1}{b(v_i^{(1)},u)}\prod_{j=1}^{n}\frac{b(u,v_j)}{b(v_j,u)} + \prod_{i=1}^{m}\frac{1}{b(u,v_j^{(1)})}\prod_{j=1}^{n}b(u,v_j)\Bigg]|v_1^{(1)},\dots,v_m^{(1)}\rangle. \tag{3.34}$$

One can get simply the first set of BE by comparing (3.34) with (3.26). Inputting $u = v_k$ in (3.26) we obtain

$$[b^{t_2}(v_k)\bar{b}(v_k)]^{\frac{N}{2}} = \prod_{i=1}^{m} b(v_i^{(1)},v_k) \qquad k = 1,2,\dots,n. \tag{3.35}$$

The second set of BE, which are the conditions of cancellation of the unwanted terms $\Lambda_k^{(1)}$ and $\tilde{\Lambda}_k^{(1)}$ are similar to the corresponding equations of the standard XXX model and can be found easily as

$$\prod_{j=1}^{n} b(v_j,v_k^{(1)}) = \prod_{i\neq k}\frac{b(v_k^{(1)},v_i^{(1)})}{b(v_i^{(1)},v_k^{(1)})} \qquad k = 1,2,\dots,m. \tag{3.36}$$

This is exactly the same equation as found in [29].

Finally, we find

$$\begin{aligned}
\nu_1(u,\{v_i\}) &= \prod_{i=1}^{n}\frac{1}{b(v_i,u)} - [b^{t_2}(u)\bar{b}(u)]^{\frac{N}{2}}\prod_{j=1}^{n}\frac{1}{b(u,v_j)}\Bigg[\prod_{i=1}^{m}\frac{1}{b(v_i^{(1)},u)}\prod_{j=1}^{n}\frac{b(u,v_j)}{b(v_j,u)} \\
&\quad + \prod_{i=1}^{m}\frac{1}{b(u,v_i^{(1)})}\prod_{j=1}^{n}b(u,v_j)\Bigg] \\
\nu_0(u,\{v_i\}) &= \prod_{i=1}^{n}\frac{1}{b(v_i,\bar{u})} - [\bar{b}^{t_2}(u)b(u)]^{\frac{N}{2}}\prod_{j=1}^{n}\frac{1}{b(\bar{u},v_j)}\Bigg[\prod_{i=1}^{m}\frac{1}{b(v_i^{(1)},\bar{u})}\prod_{j=1}^{n}\frac{b(\bar{u},v_j)}{b(v_j,\bar{u})} \\
&\quad + \prod_{i=1}^{m}\frac{1}{b(\bar{u},v_i^{(1)})}\prod_{j=1}^{n}b(\bar{u},v_j)\Bigg]
\end{aligned} \tag{3.37}$$

as the n-particle state eigenvalues of transfer matrices $\tau_1(u)$ and $\tau_0(u)$ respectively.

But the transfer matrix of our staggered model is the product of $\tau_0(u)$ and $\tau_1(u)$, therefore the eigenvalues $\nu(u,\{v_i\})$ of $\tau(u)$ are

$$\nu(u,\{v_i\}) = \nu_0(u,\{v_i\})\nu_1(u,\{v_i\}) \tag{3.38}$$

with the BEs (3.35) and (3.36) unchanged.

Let us now calculate the energy of excitations over the test 'vacuum' $|\Omega\rangle$, called bare energy, which is dressed in a real ground state due to interactions with particles in a filled Dirac sea. The bare energy is a logarithmic derivative of eigenvalues (3.37) and (3.38) at the point $u = 0$. The simple calculation gives the energy and the momentum of n-particle state $|v_1, \ldots, v_n \mid F\rangle$ as follows:

$$\begin{aligned} E_0(\{v_j\}) &= -\sum_{j=1}^{n} \left\{ \frac{1}{v_j^2 + 1/4} - \frac{1}{(v_j - \theta)^2 + 1/4} \right\} \\ \mathrm{i}P(\{v_j\}) &= \sum_{j}^{n} \left\{ \log \frac{v_j + 1/2}{v_j - 1/2} + \log \frac{v_j - \theta + 1/2}{v_j - \theta - 1/2} \right\} \end{aligned} \tag{3.39}$$

where we have redefined the spectral parameters as $v_j \to v_j - 1/2$.

The solution of the BEs (3.35) and (3.36) is usually obtained in the thermodynamic limit ($N, n, m \to \infty$, with the fixed ratio $\frac{n}{N}, \frac{m}{N}$). In this case instead of a discrete set of spectral parameters v_j one introduces the distribution of continuous density $\rho(v)$ of rapidities. The ground state is defined by filling up the Dirac sea(s) of negative energies by the electrons. It was argued in [28] that the ground state of the t–J model is defined by the string solutions of length two, which are filling of all states with negative energy. In this case from the two BEs (3.35) and (3.36) one can obtain one equation for the real part of rapidities of the strings of length two. The lowest energy value can be reached by maximal filling of negative energy states, which corresponds to $\frac{n}{N} = 1$ and with zero magnetization, corresponding to $m = \frac{n}{2}$.

In our model we have the same second set of BEs (3.36) as for ordinary t–J model, but the first set is slightly different due to shift of the spectral parameters. Therefore, the string states of length two have similar equation for the real part of rapidity. It is very natural to suppose that the strings of length two contribute to the ground state of this model too[4] but, contrary to the ordinary t–J case, as is clear from the expression of the energy (3.39), only the states with real part of the spectral parameters from the interval

$$-\infty < u < \frac{\theta}{2} \tag{3.40}$$

have to be filled in order to form a ground state. Since this is exactly equal to one-half of lattice sites N, we have a ground state corresponding to quarter filling of all states.

The thermodynamic Bethe ansatz (TBA) technique (see e.g. [3]) can now be used in order to analyse the BE for the real part of the rapidities. After taking the logarithm from the left- and right-hand sides we introduce the notation $\rho(v)$ for the density of states around v in the thermodynamic limit and obtain the integral equation for it. To investigate the low-energy excitations around the ground state we should consider in this equation rapidities only from the region $-\infty < v < \theta/2$. As a result we obtain

$$\pi\rho(v) + \int_{Q}^{\infty} \mathrm{d}u \, \frac{\rho(u)}{(v-u)^2 + 1} = \frac{1}{2} \left\{ \frac{1}{v^2 + 1/4} + \frac{1}{(v-\theta)^2 + 1/4} \right\} \tag{3.41}$$

where Q defines the spectral parameter of the Fermi level and is equal to $\theta/2$ for the ground state. As has been shown, it corresponds to quarter filling of the states.

We see that instead of the usual Fredholm equations, we obtain a Wiener-Hopf-type equation. This is the essential reason why the physics of our model differs from the one with usual half-filled ground state (for which $Q = -\infty$).

[4] Of course, in order to find a correct ground state one should analyse a full set of thermodynamic Bethe ansatz equations with strings of arbitrary length, but because these equations are even with respect to spectral parameter one can easily see that this hypothesis is reasonable.

The energy of ground state is defined by

$$E_0 = -2N \int_Q^\infty \left\{ \frac{1}{v^2+1} - \frac{1}{(v-\theta)^2+1} \right\} = -2N(\rho(0) - \rho(\theta)). \qquad (3.42)$$

It follows from (3.42), that at usual half filling $Q = -\infty$, the energy is $E = 0$ and therefore that state cannot be a ground state.

It seems to us that this model provides an interesting possibility to analyse by means of exact integrability the physics of systems with quarter-filled ground state.

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